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Vector coherent state representations, induced representations and geometric quantization: I. Scalar coherent state representations

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Abstract
Coherent state theory is shown to reproduce three categories of representations of the spectrum generating algebra for an algebraic model: (i) classical realizations which are the starting point for geometric quantization, (ii) induced unitary representations corresponding to prequantization and (iii) irreducible unitary representations obtained in geometric quantization by choice of a polarization. These representations establish an intimate relation between coherent state theory and geometric quantization in the context of induced representations.

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1. Introduction

The process of quantizing a classical system has been of interest since the early days of quantum mechanics and remains an active field of research. The most sophisticated quantization procedure is provided by *geometric quantization* (GQ). This procedure, founded on Kirillov's so-called *orbit method* [1], was initiated by Kostant [2], and Souriau [3], who extended the orbit method by capitalizing on the physical insights that come from its applications to quantum mechanics. Thus geometric quantization incorporates ideas that physicists have used for many years, but provides new avenues to address quantization of more difficult systems.

Our primary concern in this paper is the significance of GQ for the representation theory of Lie groups and Lie algebras. Its relevance to representation theory is based on the observation that quantizing a model with a spectrum generating algebra (SGA), defined in the following section, is equivalent to constructing an appropriate irreducible unitary representation of that algebra [4]. Conversely, as brought to light by GQ, the construction of a unitary irrep of a

Lie group or algebra is often equivalent to quantizing some classical Hamiltonian system. Thus, the theory of induced representations plays a central role in the quantization of a model and in quantum mechanics in general, as emphasized by Mackey [5]. Establishing an explicit correspondence between induced representation theory and GQ sheds new light on both theories.

The theory of induced representations is viewed in this paper from the perspective of coherent state theory [6–9], which incorporates other inducing constructions in terms of structures and concepts that relate naturally to quantum mechanics and GQ. It is shown that coherent state theory reproduces three categories of representations of the SGA for an algebraic model, and that these categories are related to the structures in geometric quantization. First, there are classical realizations of the SGA as functions on a phase space; GQ begins with such a classical realization. The coherent state construction also yields the induced unitary (reducible) representations that correspond to prequantization. Finally, the unitary irreducible representations, corresponding to full quantization, are obtained through means related to the choice of a polarization in GQ. These techniques are illustrated by a variety of examples.

A relationship between coherent state theory and geometric quantization follows naturally from the problem that gave rise to coherent state representations; namely, 'construct an irreducible unitary representation of a group from the properties of a set of coherent states' [8, 10]. As will be shown in the following, the expectation values of the Lie algebra over a set of coherent states give a classical representation of the Lie algebra. Thus, regaining a unitary irrep is equivalent to quantization of this classical representation. It is found that reconstructing the irrep is possible in coherent state theory only for particular orbits which give rise to the classical representations that are described as *quantizable* in geometric quantization.

It is the principal aim of this paper to show that coherent state methods provide an intuitive and practical framework for implementing the procedures of induced representations and geometric quantization. We thereby attempt to make these fundamental theories accessible to a wider community. The illustrative examples presented here are intentionally simple. However, it is noted that all three theories have been deployed in non-trivial ways. The theory of induced representations has made seminal contributions in physics, e.g., to the representation theory of the Poincaré group [11] and the space groups of crystals [12]; indeed, it is central to representation theory and quantum mechanics [5]. Geometric quantization leads to a deep understanding of the route from classical mechanics to quantum mechanics. It has been used, for example, to develop theories of quantized vortices in hydrodynamics [13] and nuclei [14] and to reproduce the irreducible unitary representations of compact Lie groups and the holomorphic discrete series of reductive Lie groups [15]. Other applications can be found in the book of Guillemin and Sternberg [16]. Similarly, coherent state theory has provided a fundamental understanding of classical-like behaviour in quantum mechanics, e.g., in the field of quantum optics [17]. It has also been used extensively [7], particularly in its vector coherent state extension [18] (outlined in the following paper), in the quantization of numerous physical models by explicit construction of the irreducible representations of their spectrum generating algebras (cf [9] for a long list of references). Our hope is that the complementary aspects of these powerful methods will result in their successful application to even more challenging problems.

It has long been known [19] that there is a close relationship between the theories of coherent states, induced representations and geometric quantization. Indeed, the Kirillov method [1, 20, 21], from which geometric quantization emerged, was expressed as an inducing construction. This relationship was used, for example, by Streater [22] to construct the irreducible unitary representations of the semidirect product oscillator group, following both the Mackey and Kirillov methods, and by Dunne [23] to construct the irreps of $SO(2, 1)$.

It is particularly well known that the coadjoint orbits of a Lie group play central roles in all three theories (cf, for example, chapter 15 of Kirillov's book [20]). The contribution of this paper is to explore the detailed relationships between the theories so that their complementary strengths may be better understood and more readily applied to problems in physics. Moreover, by establishing detailed relationships, we hope to make it easier to move between the different expressions of quantization and induced representations in the solution of complex problems.

With a relationship between scalar coherent state theory and GQ in place, the generalization to vector coherent state theory [9, 18], in which irreps of an algebra are induced from multidimensional irreps of a subalgebra, indicates new ways of applying geometric quantization to models with intrinsic degrees of freedom. Conversely, the different perspective of geometric quantization suggests possibilities for generalization of the theory of induced representations. These issues will be discussed in the sequel paper.

2. Algebraic models

An algebraic model is defined below as a model with an SGA. The quantization of an algebraic model and the construction of the irreducible unitary representations of its SGA are then related problems. However, whereas quantization starts with the classical Hamiltonian dynamics on a phase space, the theory of induced representations starts with the abstract SGA. These and other concepts invoked in the two constructions are reviewed in this section.

2.1. Observables and spectrum generating algebras

In classical mechanics, observables are realized as smooth real-valued functions on a connected phase space $\mathcal{M}$, i.e. elements of $C^\infty(\mathcal{M})$. They form an infinite-dimensional Lie algebra with Lie product given by a Poisson bracket. In quantum mechanics, observables are interpreted as Hermitian linear operators on a Hilbert space $\mathbb{H}$; they are elements of $GL(\mathbb{H})$ and form an infinite-dimensional Lie algebra with Lie product given by commutation.

The algebras $C^\infty(\mathcal{M})$ and $GL(\mathbb{H})$ for a given physical system are different [24]. However, a simple relationship may be established between finite-dimensional subalgebras of $C^\infty(\mathcal{M})$ and $GL(\mathbb{H})$. Thus, it is convenient to consider an abstract Lie algebra of observables $\mathfrak{g}$ that is real and finite dimensional and can be represented classically by a homomorphism $J : \mathfrak{g} \to C^\infty(\mathcal{M})$ and quantum mechanically by a unitary representation $T : \mathfrak{g} \to GL(\mathbb{H})$. Let $\mathcal{A} = J(A)$ and $\hat{A} = T(A)$ denote classical and quantal representations, respectively, of an element $A \in \mathfrak{g}$.

Then, if elements A, B and $C \in \mathfrak{g}$ satisfy the commutation relations

$$[A, B] = \mathrm{i}\hbar C \tag{1}$$

the corresponding linear operators and functions satisfy

$$[\hat{A}, \hat{B}] = \mathrm{i}\hbar\hat{C} \tag{2}$$

and

$$\{\mathcal{A}, \mathcal{B}\} = \mathcal{C} \tag{3}$$

where $\{\,,\}$ denotes the classical Poisson bracket[4]. (More precisely, the classical homomorphism is given by $A \to \mathrm{i}\hbar\mathcal{A}$, so that $\{(\mathrm{i}\hbar\mathcal{A}), (\mathrm{i}\hbar\mathcal{B})\} = \mathrm{i}\hbar(\mathrm{i}\hbar\mathcal{C})$.)

[4] Note that we follow the practice, common in quantum mechanics, of representing the infinitesimal generators of a unitary group representation, which correspond to physical observables, by Hermitian operators. To regard the real linear span of such operators as a real Lie algebra then requires inclusion of a factor i in the commutation relations. The Poisson bracket needs no such factor.

Let $\mathcal{G} \subset C^{\infty}(\mathcal{M})$ denote the classical algebra $\mathcal{G} = \{J(A)|A \in \mathfrak{g}\}$. We shall describe the algebra $\mathcal{G}$ (and hence the algebra $\mathfrak{g}$) as a *spectrum generating algebra* for the classical system if the values of the observables in $\mathcal{G}$ are sufficient to uniquely specify a point in $\mathcal{M}$ and their gradients span the tangent space of $\mathcal{M}$ at every point. (Other compatible definitions of an SGA can be found in the literature [25]. For example, a classical SGA can be defined by requiring that the only functions in $C^{\infty}(\mathcal{M})$ that Poisson commute with all elements of $\mathcal{G}$ are the constant functions [15].)

A finite-dimensional Lie algebra $\mathfrak{g}$ is said to be an SGA for a quantal system if the Hilbert space for the system carries an irreducible representation of $\mathfrak{g}$. However, to be useful, one may also require that the Hamiltonian and other important observables of the system should be simply expressible in terms of $\mathfrak{g}$, e.g., by belonging to its universal enveloping algebra. Thus, in quantizing a classical model, we shall seek a quantal system with the same SGA as the classical model. A model dynamical system that has a finite-dimensional SGA is said to be an *algebraic model*.

2.2. Phase spaces as coadjoint orbits

Let G be a group of canonical transformations (i.e. symplectomorphisms) of a classical phase space $\mathcal{M}$ for a model. Then, if G acts transitively on $\mathcal{M}$, it is said to be a *dynamical group* for the model. If an element $g \in G$ sends a point $m \in \mathcal{M}$ to $m \cdot g \in \mathcal{M}$, then $\mathcal{M}$ is the group orbit

$$\mathcal{M} = \{m \cdot g \mid g \in G\} \tag{4}$$

and diffeomorphic to the factor space $H_m \backslash G$ with isotropy subgroup

$$H_m = \{h \in g \mid m \cdot h = m\}. \tag{5}$$

A remarkable fact [2, 3] that will be used extensively in the following is that a phase space with a dynamical group can be identified with a coadjoint orbit. Conversely, every coadjoint orbit is a phase space. Moreover, the Lie algebra $\mathfrak{g}$ of G is an SGA for the model. Note that in quantum mechanics, one is often interested in projective representations of a given dynamical group. Thus, we consider projective as well as true representations; in practice, it is often simpler to choose a dynamical group that is simply connected so that all its representations are true representations.

Recall [26] that G has a natural adjoint action on its Lie algebra $\mathfrak{g}$

$$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g} \qquad A \mapsto A(g) = \mathrm{Ad}(g)A \tag{6}$$

where, for a matrix group, $\mathrm{Ad}(g)A = gAg^{-1}$. G also has a coadjoint action on the space $\mathfrak{g}^*$ of real-valued linear functionals on $\mathfrak{g}$ (the dual of $\mathfrak{g}$). The action of an element $\rho \in \mathfrak{g}^*$ on an element $A \in \mathfrak{g}$ is given by the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$, expressed as $\rho(A) = \mathrm{Tr}(\rho A)$ for matrices. The coadjoint action is then defined, for $\rho \in \mathfrak{g}^*$ and $g \in G$, by $\rho \to \rho_g$, where

$$\rho_g(A) = \rho(A(g)) \qquad \forall A \in \mathfrak{g}. \tag{7}$$

Thus, the coadjoint orbit

$$\mathcal{O}_\rho = \{\rho_g \mid g \in G\} \tag{8}$$

is diffeomorphic to the factor space $H_\rho \backslash G$ with isotropy subgroup $H_\rho = \{h \in g \mid \rho_h = \rho\}$. We shall refer to an element ρ of $\mathfrak{g}^*$ as a *density*. Now if a density $\rho \in \mathfrak{g}^*$ is chosen such that $H_\rho = H_m$ then there is a diffeomorphism $\mathcal{M} \to \mathcal{O}_\rho$ in which $m \mapsto \rho$ and $m \cdot g \mapsto \rho_g$. This map is known as a *moment map* [2, 3, 26].

Such a moment map defines a classical representation $J : \mathfrak{g} \to \mathcal{G}$; $A \mapsto \mathcal{A} = J(A)$ of the Lie algebra $\mathfrak{g}$ as functions over the classical phase space $\mathcal{M}$, defined by

$$\mathcal{A}(m \cdot g) = \rho_g(A) \tag{9}$$

with Poisson bracket given by

$$\{\mathcal{A}, \mathcal{B}\} = \omega(A, B) \tag{10}$$

where ω is the antisymmetric two-form[5] with values at $m \cdot g \in \mathcal{M}$ given by

$$\omega_{m\cdot g}(A, B) = -\frac{\mathrm{i}}{\hbar}\rho_g([A, B]) \qquad \forall A, B \in \mathfrak{g}. \tag{11}$$

Thus, the moment map $\mathcal{M} \to \mathcal{O}_\rho$ defines a symplectic form ω on $\mathcal{O}_\rho$. This form is known to be non-degenerate and the map $\mathcal{M} \to \mathcal{O}_m$ is a symplectomorphism.

2.3. Geometric quantization

Geometric quantization formalizes the ideas of Dirac [27] concerning the quantization of a classical system. The idea is to replace the classical (Poisson bracket) algebra of functions on a phase space by a unitary representation, i.e. map each classical observable (function) $\mathcal{F}$ to a Hermitian operator $\hat{F}$ such that if

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \mathcal{F}_3 \tag{12}$$

then

$$[\hat{F}_1, \hat{F}_2] = \mathrm{i}\hbar\hat{F}_3. \tag{13}$$

There are some additional requirements of the *Dirac map*. First, all constant functions must map to multiples of the identity operator $\hat{I}$. Second, the unitary representation should be irreducible.

GQ solves the first part of the Dirac problem by a construction known as prequantization. However, it is now known that there are generally no irreducible unitary representations of the full algebra of observables [24]. Thus, a complete solution of the Dirac problem is impossible. When there exists a finite-dimensional SGA for the model then, under certain quantizability conditions, the restriction to the SGA of the representation given by prequantization becomes fully reducible. Moreover, GQ gives a prescription for the reduction in terms of a *polarization*. For details of the techniques of GQ, see Woodhouse [28].

It will be shown in the following that the reducible representation of an SGA given by prequantization of a classical phase space diffeomorphic to $H_\rho\backslash G$ integrates to the reducible representation of G induced from a one-dimensional representation of H_ρ in the standard theory of induced representations. Likewise, the irreducible representations obtained by introducing a polarization are obtainable by the coherent state inducing construction.

3. Scalar coherent state representations

In this section, we show how the coherent state construction reproduces the three categories of representations of the SGA of an algebraic model: classical realizations, reducible unitary representations corresponding to prequantization and irreducible unitary representations of a full quantization.

Let G with Lie algebra $\mathfrak{g}$ denote the dynamical group of an algebraic model and let T denote an abstract (possibly projective) unitary representation of G on a Hilbert space $\mathbb{H}$. There is no need to make a precise specification of T. For example, if G has a right-invariant measure $\mathrm{d}v(g)$, $\mathbb{H}$ could be the space $\mathcal{L}^2(G)$ of square-integrable functions with respect to this measure and T the regular representation. Or, for application to models of many-particle systems, $\mathbb{H}$ might be the standard many-particle Hilbert space $\mathcal{L}^2(\mathbb{R}^{3N})$ of square-integrable functions

[5] With the realization of the Lie algebra $\mathfrak{g}$ as a set of invariant vector fields on $\mathcal{M}$, ω becomes a two-form on $\mathcal{M}$.

of many-particle Cartesian coordinates and T a Weil [29] or Schrödinger representation (see section 4.1).

For notational convenience we denote the representation $T(A)$ of an element $A \in \mathfrak{g}$ by $\hat{A} \equiv T(A)$.

3.1. Classical realizations of $\mathfrak{g}$

In section 2.2, the classical phase space for an algebraic model is shown to be expressible as a coadjoint orbit. In this section, we show that these coadjoint orbits can be projected from group orbits in the Hilbert space.

For any state $|0\rangle \in \mathbb{H}$ of unit norm (i.e. $\langle 0|0\rangle = 1$) there is a system of coherent states [6]

$$\{|g\rangle = T(g^{-1})|0\rangle;\ g \in G\} \tag{14}$$

and a corresponding set of dual states

$$\{\langle g| = \langle 0|T(g);\ g \in G\}. \tag{15}$$

Moreover, there is a natural identification of any state with a density, i.e. an element of the dual algebra $\mathfrak{g}^*$. Thus, the coherent states define a system of densities

$$\rho(A) = \langle 0|\hat{A}|0\rangle \tag{16}$$

$$\rho_g(A) = \langle g|\hat{A}|g\rangle = \langle 0|\hat{A}(g)|0\rangle = \rho(A(g)) \qquad g \in G \tag{17}$$

where

$$\hat{A}(g) = T(\mathrm{Ad}(g)A) = T(g)\hat{A}T(g^{-1}) \qquad A \in \mathfrak{g}. \tag{18}$$

It follows that the coherent states determine a coadjoint orbit

$$\mathcal{O}_\rho = \{\rho_g;\ g \in G\}. \tag{19}$$

They also determine a map J from the Lie algebra $\mathfrak{g}$ (cf section 2.2) to functions on $\mathcal{O}_\rho$ in which an element $A \in \mathfrak{g}$ is mapped to a function $\mathcal{A} = J(A)$ with values

$$\mathcal{A}(g) = \langle g|\hat{A}|g\rangle = \rho_g(A). \tag{20}$$

The map J is a classical representation of $\mathfrak{g}$. For, if $[\hat{A}, \hat{B}] = \mathrm{i}\hbar\hat{C}$, then the corresponding functions $\mathcal{A} = J(A)$, $\mathcal{B} = J(B)$ and $\mathcal{C} = J(C)$ satisfy a Poisson bracket relationship defined by

$$\{\mathcal{A}, \mathcal{B}\}(g) = -\frac{\mathrm{i}}{\hbar}\langle g|[\hat{A}, \hat{B}]|g\rangle = \mathcal{C}(g). \tag{21}$$

The Poisson bracket can be expressed in terms of local coordinates for $\mathcal{O}_\rho$ as follows. First observe that $\mathcal{O}_\rho$ is diffeomorphic to the coset space $H_\rho\backslash G$, where

$$H_\rho = \{h \in G | \rho_h = \rho\}. \tag{22}$$

Then the Lie algebra $\mathfrak{h}_\rho$ of the stability subgroup H_ρ is the set

$$\mathfrak{h}_\rho = \{X \in \mathfrak{g} \mid \rho([X, A]) = 0, \forall A \in \mathfrak{g}\}. \tag{23}$$

Thus, if $\{A_i\}$ is a basis for $\mathfrak{h}_\rho$ and $\{A_\nu\}$ completes a basis for $\mathfrak{g}$, coordinates are defined for elements of G about a point $g \in G$ by setting

$$g(\xi, x) = \exp\left(-\frac{\mathrm{i}}{\hbar}\sum_{\mathrm{i}} \xi^i A_i\right) \exp\left(-\frac{\mathrm{i}}{\hbar}\sum_{\nu} x^\nu A_\nu\right) g. \tag{24}$$

It follows that at $g = g(0,0)$,

$$\frac{\partial \mathcal{A}(g)}{\partial \xi^i} \equiv \frac{\partial \mathcal{A}(g(\xi,x))}{\partial \xi^i}\Big|_{\xi=x=0} = -\frac{\mathrm{i}}{\hbar}\langle 0|[\hat{A}_i, \hat{A}(g)]|0\rangle = 0. \tag{25}$$

However, the corresponding derivatives with respect to the $\{x^\nu\}$ coordinates do not, in general, vanish. Thus, the set $\{x^\nu\}$ serves as local coordinates for $\mathcal{O}_\rho \sim H_\rho\backslash G$. With respect to these coordinates, we then define

$$(\partial_\mu \mathcal{A})(g) \equiv \frac{\partial \mathcal{A}(g(\xi,x))}{\partial x^\mu}\Big|_{\xi=x=0} = \sum_\nu \omega_{\mu\nu} A^\nu(g) \tag{26}$$

where $A^\nu(g)$ is a coefficient in the expansion

$$A(g) = \sum_i A^i(g) A_i + \sum_\nu A^\nu(g) A_\nu \tag{27}$$

and

$$\omega_{\mu\nu} = -\frac{\mathrm{i}}{\hbar}\langle 0|[\hat{A}_\mu, \hat{A}_\nu]|0\rangle. \tag{28}$$

The Poisson bracket of equation (21) is then expressed in the familiar form

$$\{\mathcal{A},\mathcal{B}\}(g) = \sum_{\mu\nu} A^\mu(g)\omega_{\mu\nu} B^\nu(g) = \sum_{\mu\nu} (\partial_\mu \mathcal{A})(g)\omega^{\mu\nu}(\partial_\nu \mathcal{B})(g) \tag{29}$$

where the matrix $(\omega^{\mu\nu})$ is defined such that

$$\sum_\nu \omega^{\mu\nu}\omega_{\lambda\nu} = \delta^\mu_\lambda. \tag{30}$$

In this form, the Poisson bracket can be extended to all C^∞ functions on $\mathcal{O}_\rho$.

It will be noted that the above results are expressed in terms of a specific choice of the basis elements $\{A_\nu\}$ of the Lie algebra. Thus, it is important to ask if the results depend on the choice. Two distinct kinds of transformations of the basis are possible:

$$A_\nu \to \sum_\mu A_\mu \gamma_{\mu\nu}. \tag{31}$$

and

$$A_\nu \to A_\nu + \sum_i A_i \gamma_{i\nu}. \tag{32}$$

Combinations of the two kinds are also possible but it is instructive to consider them separately. Transformations of the first kind leave the subspace of the Lie algebra spanned by the elements $\{A_\nu\}$ invariant. They generate normal coordinate transformations on $\mathcal{O}_\rho$. However, all physical expressions, such as the values of observables and their Poisson brackets, are manifestly covariant relative to such coordinate transformation. Transformations of the second kind, corresponding to changes of the linear span of the $\{A_\nu\}$ basis, are known as *gauge transformations*. Because of the definition of the intrinsic subalgebra, the symplectic form $\omega = \sum_{\mu\nu}\omega_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ is seen to be invariant under a gauge transformation. Moreover, from definition (26), it follows that

$$\mathrm{i}\hbar(\partial_\nu \mathcal{A})(g) = \langle 0|[\hat{A}_\nu, \hat{A}(g)]|0\rangle. \tag{33}$$

Thus, it follows that $(\partial_\nu \mathcal{A})(g)$ and hence the Poisson bracket are also gauge invariant.

Note that different classical representations result from different choices of the state $|0\rangle$, even in the case when the representation T is irreducible.

3.2. Induced representations

Given an abstract unitary representation T of the dynamical group G over a Hilbert space $\mathbb{H}$, a coherent state representation Γ is defined by the specification of a functional $\langle\varphi|$ on a G-invariant dense subspace $\mathbb{H}_D \subset \mathbb{H}$. A state $|\psi\rangle \in \mathbb{H}_D$ then has coherent state wavefunction ψ defined over G by

$$\psi(g) = \langle\varphi|T(g)|\psi\rangle \qquad \forall g \in G. \tag{34}$$

A Hilbert space $\mathcal{H}$ for a coherent state representation Γ is the completion of the space of such coherent state wavefunctions with respect to the inner product

$$(\psi, \psi') = \langle\psi|\psi'\rangle \tag{35}$$

where the inner product on the right is that of $\mathbb{H}$. The coherent state representation Γ is then defined by

$$[\Gamma(g)\psi](g') = \psi(g'g). \tag{36}$$

Clearly there are many coherent state representations depending on the choice of the functional $\langle\varphi|$ and Hilbert space $\mathbb{H}$. If the functional $\langle\varphi|$ is chosen such that

$$\psi(hg) = \langle\varphi|T(h)T(g)|\psi\rangle = \chi(h)\psi(g) \qquad \forall\, h \in H_\rho \tag{37}$$

where χ is a one-dimensional representation of the subgroup $H_\rho \subset G$, we say that the coherent state representation Γ is induced from the representation χ of H_ρ.

If G is compact, the space $\mathcal{H}$ of coherent state wavefunctions is contained in $\mathcal{L}^2(G)$ and Γ is isomorphic to a subrepresentation of the regular representation [8] (this can be seen from the inner product given in section 3.4). More generally, if $\langle\varphi|$ is such that $\mathcal{H}$ consists of all functions which satisfy equation (37) and whose absolute values are in $\mathcal{L}^2(H_\rho\backslash G)$, then Γ is said to be a *standard* (Mackey) induced representation [5]. In general, the representation Γ is highly reducible. It will be shown in section 3.3 that, by imposing extra conditions on the choice of the functional $\langle\varphi|$, it is possible to proceed directly to the irreducible representations of quantization.

The above coherent state wavefunctions are defined as functions over the group G. However, in practical applications, it is generally more useful to represent them as functions over a suitable set of $H_\rho\backslash G$ coset representatives [8]. Recall that a set of coset representatives $K = \{k(g) \in H_\rho g;\, g \in G\}$ defines a unique factorization $g = h(g)k(g)$, with $h(g) \in H_\rho$, of every $g \in G$. Hence, it follows from identity (37) that the restriction of $\psi \in \mathcal{H}$ to the subset $K \subset G$ is sufficient to uniquely define ψ. Often it is also convenient to consider factorizations of the type $g = h(g)k(g)$ with $h(g) \in H_\rho^c$ and $k(g) \in K$, where K is a subset of $H_\rho^c\backslash G^c$ coset representatives and H_ρ^c and G^c are the complex extensions of H_ρ and G, respectively. Note that identity (37) does not require that the wavefunctions are functions on $H_\rho\backslash G$; in general, ψ need only be a section of a complex line bundle associated with the principal bundle $G \rightarrow H_\rho\backslash G$. The Hilbert space $\mathcal{H}$, then, can be viewed as a space of sections of a complex line bundle over $H_\rho\backslash G$ [30].

We now show that, if a representation Γ is induced from a representation χ of H_ρ having the property that

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\chi(\mathrm{e}^{-\mathrm{i}At})\Big|_{t=0} = \chi(A) \equiv \rho(A) \qquad A \in \mathfrak{h}_\rho \tag{38}$$

then the corresponding representation of the Lie algebra $\mathfrak{g}$, defined by

$$[\Gamma(A)\psi](g) = \langle\varphi|T(g)\hat{A}|\psi\rangle = \psi(gA) \qquad A \in \mathfrak{g} \tag{39}$$

where

$$\psi(gA) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(g\,\mathrm{e}^{-\mathrm{i}tA})\Big|_{t=0} \tag{40}$$

is a subrepresentation of that given by prequantization. (Such a relationship was shown in more restricted contexts, for example, by Dunne [23] and Rawnsley [19].)

Note, however, that for this prequantization to be possible, the representation χ of $\mathfrak{h}_\rho$ defined by ρ must be a subrepresentation of the restriction to $\mathfrak{h}_\rho \subset \mathfrak{g}$ of some unitary representation T of $\mathfrak{g}$. When this condition is satisfied, we say (in the language of geometric quantization) that the classical representation of $\mathfrak{g}$ defined by ρ is *quantizable*.

First observe that

$$\psi(gA) = \psi(A(g)g) \tag{41}$$

where $A(g) = \mathrm{Ad}_g(A)$. Substitution of the identities

$$\psi(A_i g) = \chi(A_i)\psi(g) \qquad \psi(A_\nu g) = \mathrm{i}\hbar(\partial_\nu\psi)(g) \tag{42}$$

where

$$\mathrm{i}\hbar(\partial_\nu\psi)(g) = \mathrm{i}\hbar\frac{\partial}{\partial x^\nu}\psi\left(\exp\left(-\frac{\mathrm{i}}{\hbar}\sum_\nu x^\nu A_\nu\right)g\right)\Bigg|_{x=0} \tag{43}$$

into expansion (27) of $A(g)$, then gives the explicit expression

$$[\Gamma(A)\psi](g) = \sum_i A^i(g)\chi(A_i)\psi(g) + \mathrm{i}\hbar\sum_\nu A^\nu(g)(\partial_\nu\psi)(g) \tag{44}$$

for the action of $\Gamma(A)$ on coherent state wavefunctions.

Expression (44) of Γ depends on expansion (27) of $A(g)$. Thus it is coordinate dependent and gauge dependent. However, it can be expressed in a covariant form by taking advantage of the symplectic structure of the classical phase space. By expanding the classical representation $\mathcal{A}(g) = \rho(A(g))$ of $A \in \mathfrak{g}$,

$$\mathcal{A}(g) = \sum_i A^i(g)\chi(A_i) + \sum_\nu A^\nu(g)\rho(A_\nu) \tag{45}$$

equation (44) becomes

$$[\Gamma(A)\psi](g) = \mathcal{A}(g)\psi(g) + \mathrm{i}\hbar\sum_\nu A^\nu(g)(\nabla_\nu\psi)(g) \tag{46}$$

where

$$\mathrm{i}\hbar(\nabla_\nu\psi)(g) = \mathrm{i}\hbar(\partial_\nu\psi)(g) - \rho(A_\nu)\psi(g). \tag{47}$$

The first term, $\mathcal{A}(g)\psi(g)$, of equation (46) is manifestly covariant. Moreover, from the definition, $[\Gamma(A)\psi](g) = \psi(A(g)g)$, it follows that

$$\mathrm{i}\hbar\sum_\nu A^\nu(g)(\nabla_\nu\psi)(g) = \psi(A(g)g) - \rho(A(g))\psi(g). \tag{48}$$

Thus, the second term is also covariant. We shall refer to the operator $\nabla_{X_\mathcal{A}} = \sum_\nu A^\nu\nabla_\nu$, defined such that

$$[\nabla_{X_\mathcal{A}}\psi](g) = \sum_\nu A^\nu(g)(\nabla_\nu\psi)(g) \tag{49}$$

as a covariant derivative, in accordance with standard terminology. The induced representation $\Gamma(A)$ of an arbitrary element $A \in \mathfrak{g}$ is then expressed in the covariant form

$$\Gamma(A) = \mathcal{A} + \mathrm{i}\hbar\nabla_{X_\mathcal{A}}. \tag{50}$$

A similar expression can also be derived within the framework of Berezin quantization [19].

It is shown in the appendix that, for a particular choice of gauge, $\nabla_{X_{\mathcal{A}}}$ is expressible as a sum

$$\nabla_{X_{\mathcal{A}}} = X_{\mathcal{A}} + \frac{\mathrm{i}}{\hbar}\theta(X_{\mathcal{A}}) \tag{51}$$

where $X_{\mathcal{A}}$ is a Hamiltonian vector field defined by the classical function $\mathcal{A}$ and θ is a one-form (gauge potential). It is also shown that the symplectic two-form ω of the manifold $\mathcal{O}_\rho$ is the exterior derivative

$$\omega = \mathrm{d}\theta. \tag{52}$$

Thus, θ is a symplectic potential for $\mathcal{O}_\rho$. Note that the values of the symplectic potential θ depend on the choice of $\{A_\nu\}$ (cf equation (24)). As a consequence, θ is only defined to within a gauge transformation $\theta \to \theta + \mathrm{d}\alpha$. However, its exterior derivative is gauge independent.

The coherent state representation $\Gamma : A \to \mathcal{A} + \mathrm{i}\hbar\nabla_{X_{\mathcal{A}}}$ is now observed to be of the standard form of prequantization in the theory of geometric quantization. Thus, prequantization of an algebraic model is equivalent to a standard (Mackey) representation induced from a one-dimensional irrep of a suitable subgroup. However, depending on the choice of functional $\langle\varphi|$ and starting Hilbert space $\mathbb{H}$, the Hilbert space $\mathcal{H}$ of a coherent state representation may be an invariant subspace of that of prequantization. Indeed, as we show in the following section, it is often possible to choose the functional $\langle\varphi|$ such that the coherent state representation is irreducible.

Geometric quantization shows that prequantization can be extended to the whole infinite-dimensional classical algebra of all functions on the phase space. However, the extension is not fully reducible and, as presently formulated, does not apply generally to an arbitrary coherent state representation, i.e. a coherent state representation that is not equivalent to a standard induced representation. The theory of induced representations also extends, albeit in different ways. For example, it is possible to induce coherent state representations from a representation of a subgroup $H \subset G$ for which $H\backslash G$ is not symplectic. It is also possible to induce representations that are not unitary. Some of the possibilities will be illustrated with examples in section 4.

3.3. Irreducible representations

The full quantization of an algebraic model corresponds to the construction of an irreducible unitary representation of its SGA. The usefulness of scalar coherent state induction, and indeed the full VCS theory, resides in its facility to construct such representations in a practical and computationally tractable manner. All that is needed is a functional $\langle\varphi|$ that uniquely characterizes an irrep. Such a functional can often be defined, for example, by extending condition (37) to a suitable subgroup P in the chain $H_\rho \subset P \subset G^c$, where G^c is the complex extension of G. In making the extension from a real subgroup $H_\rho \subset G$ to a complex subgroup $P \subset G^c$, we recall that a similar extension [22, 23], applied to Kirillov's orbit method, resulted in a method for constructing certain irreducible representations for semisimple Lie groups.

In making this extension, a technical concern is that, whereas a representation of an SGA $\mathfrak{g}$ extends linearly to the complex extension $\mathfrak{g}^c$ of $\mathfrak{g}$, the corresponding extension of the generic unitary representation T of the real group G may not converge for all of G^c. However, it is sufficient for the purpose of defining an irreducible coherent state representation if the action of T on some dense subspace $\mathbb{H}_D \subseteq \mathbb{H}$ can be extended to a suitable subset $U(P) \subset P$ of a subgroup $P \subset G^c$ which contains H_ρ. Let $\tilde{\chi}$ denote a one-dimensional irrep of $P \subset G^c$ which restricts to a unitary irrep χ of $H_\rho \subset P$. Now suppose a functional $\langle\varphi|$ is chosen such that

$$\langle\varphi|T(z)T(g)|\psi\rangle = \tilde{\chi}(z)\psi(g) \qquad \forall z \in U(P). \tag{53}$$

It will be shown by examples in the following sections that, for many categories of groups, there are natural choices of P and its representation $\tilde{\chi}$ for which the corresponding coherent state representation is irreducible.

Subgroups satisfying these conditions are familiar in the holomorphic induction of irreducible representations [31]. For example, if G were semisimple and the isotropy subgroup $H_\rho \subset G$ for the coadjoint orbit $H_\rho\backslash G$, as defined above, were a Cartan subgroup, then a suitable subgroup $P \subset G^c$ would be the Borel subgroup generated by H_ρ and the exponentials of a set of raising (or lowering) operators. A suitable one-dimensional representation of P would then be defined by a dominant integral highest weight for a unitary irrep of G. More generally, if H_ρ were a Levi subgroup, P would be parabolic. Non-unitary irreps can also be induced in this way. However, they are not normally described as quantizations.

The construction outlined above is a generalization of holomorphic induction and, for convenience, in this situation we will speak of 'the representation of G induced from a representation of a subgroup $P \subset G^c$'.

Apart from imposing the stronger condition (53), the coherent state construction is the same as in section 3.2. However, the stronger condition restricts the set of coherent state wavefunctions to a subset with the result that the coherent state representation becomes an irreducible subrepresentation of that given by prequantization.

Now if a unitary coherent state representation Γ of a dynamical group G induced from a representation $\tilde{\chi}$ of a subgroup $P \subset G^c$ defines an irreducible representation of the Lie algebra $\mathfrak{g}$ and if the representation $\tilde{\chi}$ satisfies the equality

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\chi}(\mathrm{e}^{-\mathrm{i}At})\Big|_{t=0} = \tilde{\chi}(A) \equiv \rho(A) \qquad A \in \mathfrak{p} \tag{54}$$

then we say that Γ is a quantization of the classical representation of $\mathfrak{g}$ defined by ρ.

Note, however, that for this quantization to be possible the representation χ of $\mathfrak{h}$ must extend to a representation $\tilde{\chi}$ of a subalgebra $\mathfrak{p} \subset \mathfrak{g}^c$ which is contained in a unique irrep of $\mathfrak{g}^c$ which restricts to a unitary irrep of $\mathfrak{g}$.

In the theory of geometric quantization, one would say that the choice of subgroup $P \subset G^c$ defines an *invariant polarization* [2, 32]. Recall that a basis for $\mathfrak{h}^c\backslash\mathfrak{g}^c$ defines a basis for the complex extension of the tangent space at every point of the classical phase space $H_\rho\backslash G$. A polarization provides a separation of the tangent space at each point of this phase space into canonical space-like and momentum-like subspaces.

Let $\mathfrak{p}$ denote the Lie algebra of P. According to Woodhouse [28], the subalgebra $\mathfrak{p} \subset \mathfrak{g}^c$ generates an invariant polarization if it satisfies the conditions:

(i) $\rho([A, B]) = 0$ for any $A, B \in \mathfrak{p}$,
(ii) $\dim_{\mathbb{R}} \mathfrak{g} + \dim_{\mathbb{R}} \mathfrak{h}_\rho = 2 \dim_{\mathbb{C}} \mathfrak{p}$,
(iii) $\mathfrak{p}$ is invariant under the adjoint action of H_ρ.

The first condition ensures that the polarization is isotropic, i.e. contains no canonically conjugate observables. The second condition ensures that $\mathfrak{p}$ is a maximal subalgebra for which the first condition holds; the polarization is then said to be Lagrangian on $H_\rho\backslash G$. This condition ensures that $\mathfrak{p}$ is sufficiently large that a representation of the group P characterizes an irrep of G. The final condition ensures that the polarization is well defined on $H_\rho\backslash G$. These conditions extend the definition of a parabolic subalgebra for a semisimple Lie algebra to the general situation.

As we illustrate with several examples in section 4, the choice of a suitable subgroup $P \subset G^c$ for a coherent state quantization also defines an invariant polarization according to the above criteria.

3.4. Coherent state inner products

For a coherent state irrep that belongs to the discrete series, an inner product is defined in the following standard way. For a given reference state $|\varphi\rangle$, let $\mathbb{I}$ denote the integral

$$\mathbb{I} = \int T(g^{-1})|\varphi\rangle\langle\varphi|T(g)\,\mathrm{d}v(g) \tag{55}$$

where $\mathrm{d}v$ is a right-invariant measure on G. This integral converges if $|\varphi\rangle$ is a normalizable state vector in an irreducible subspace of T that carries a discrete series representation. The integral then defines $\mathbb{I}$ as a well-defined operator on the Hilbert space. Moreover, it commutes with the representation $T(g')$ of any element $g' \in G$. Hence, by Schur's lemma, $\mathbb{I}$ is a multiple of the identity on the irreducible subspace containing the vector $|\varphi\rangle$. It follows that the space of coherent state wavefunctions

$$\mathcal{H} = \{\psi \,|\, \psi(g) = \langle\varphi|T(g)|\psi\rangle, |\psi\rangle \in \mathbb{H}\}, \tag{56}$$

where $\mathbb{H}$ is the Hilbert space for the representation T, has inner product given to within a convenient norm factor by

$$(\psi, \psi') = \int \psi^*(g)\psi'(g)\,\mathrm{d}v(g). \tag{57}$$

If the representation χ of H_ρ is unitary, the coherent state wavefunctions have the property

$$\psi^*(hg)\psi'(hg) = \psi^*(g)\psi'(g). \tag{58}$$

Then the integral over the group in equation (57) can be restricted to an integral over the coset space $H_\rho\backslash G$ with a right-invariant measure induced from that on G.

When $\langle\varphi|$ is a functional on a dense subspace $\mathbb{H}_D$ of $\mathbb{H}$, the integral $\mathbb{I}$ may not converge. However, the corresponding integral over $H_\rho\backslash G$ may converge and, if so, it is sufficient to define an inner product of coherent state wavefunctions in parallel with Mackey's construction of inner products for induced representations of semidirect product groups.

Inner products for more general coherent state representations are constructed by K-matrix methods [33] and the related integral methods of Rowe and Repka [34].

4. Examples

Examples are given in the following to illustrate systematic procedures for carrying out the prescriptions of geometric quantization within the framework of coherent state representation theory. The first example, for the nilpotent Heisenberg–Weyl (HW) algebra, serves as a useful prototype for more general applications. The familiar quantizations of this algebra known as Schrödinger quantization and the Bargmann–Segal representation [35] are both illustrated. The second and third examples are prototypes of semisimple and semidirect sum Lie algebras, respectively. The examples show that coherent state theory provides simple and natural routes through the (sometimes subtle) methods of geometric quantization. Often there is more than one path. There may be a choice of polarization (as illustrated by two representations, one real and one holomorphic, for the HW algebra) and a choice of the functional form of the resulting Hilbert space (illustrated for $SU(2)$).

4.1. The nilpotent Heisenberg–Weyl (HW) algebra

A generic unitary representation T of the HW algebra is spanned by Hermitian operators, $\{\hat{q}, \hat{p}, \hat{I}\}$, on a Hilbert $\mathbb{H}$ with commutation relations

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar\hat{I} \qquad [\hat{q}, \hat{I}] = 0 \qquad [\hat{p}, \hat{I}] = 0. \tag{59}$$

The representations of the so-called oscillator group, which contains the Heisenberg–Weyl group as a normal subgroup, were constructed by Streater [22] using both Mackey and Kirillov methods.

4.1.1. Schrödinger quantization. Let $|0\rangle \in \mathbb{H}$ denote any normalized state for which

$$\langle 0|\hat{I}|0\rangle = 1 \qquad \langle 0|\hat{q}|0\rangle = \langle 0|\hat{p}|0\rangle = 0. \tag{60}$$

If elements of the HW group are parametrized

$$T(g(\theta, q, p)) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{I}}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}q\hat{p}} \tag{61}$$

the group conjugates of $\{\hat{q}, \hat{p}, \hat{I}\}$ are

$$\begin{aligned} \hat{q}(g) &= T(g)\hat{q}T(g^{-1}) = \hat{q} + q\hat{I} \\ \hat{p}(g) &= T(g)\hat{p}T(g^{-1}) = \hat{p} + p\hat{I} \\ \hat{I}(g) &= T(g)\hat{I}T(g^{-1}) = \hat{I} \end{aligned} \tag{62}$$

and a classical realization of the HW algebra is given by the functions $\{\mathcal{Q}, \mathcal{P}, \mathcal{I}\}$ of p and q with

$$\begin{aligned} \mathcal{Q}(p, q) &= \langle 0|\hat{q}(g)|0\rangle = q \\ \mathcal{P}(p, q) &= \langle 0|\hat{p}(g)|0\rangle = p \\ \mathcal{I}(p, q) &= \langle 0|\hat{I}(g)|0\rangle = 1. \end{aligned} \tag{63}$$

The Poisson bracket of $\mathcal{Q}$ and $\mathcal{P}$, for example, is given by

$$\{\mathcal{Q}, \mathcal{P}\}(p, q) = -\frac{\mathrm{i}}{\hbar}\langle 0|[\hat{q}(g), \hat{p}(g)]|0\rangle = \mathcal{I}(p, q). \tag{64}$$

An induced representation of the HW algebra, equivalent to a prequantization, is now constructed by coherent state techniques as follows. (Note that we employ a different coordinate chart than used in section 3.2; although equivalent results are obtained in any chart, the coordinates used here are standard for this example.) Choosing $\langle\varphi| = \langle 0|$ to be some normalized state satisfying equation (60), as above, and factoring out the phases generated by the identity $\hat{I}$, a state $|\psi\rangle$ of a model with the HW algebra as its SGA is assigned a coherent state wavefunction ψ defined over the classical phase space (the p–q plane) by

$$\psi(p, q) = \langle 0|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}q\hat{p}}|\psi\rangle. \tag{65}$$

The corresponding coherent state representation Γ of an element $\hat{A}$ of the Heisenberg–Weyl algebra, defined generally by

$$\begin{aligned} [\Gamma(\hat{A})\psi](p, q) &= \langle 0|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}q\hat{p}}\hat{A}|\psi\rangle \\ &= \langle 0|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\left(\hat{A} + \frac{\mathrm{i}}{\hbar}q[\hat{p}, \hat{A}]\right)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}q\hat{p}}|\psi\rangle \end{aligned} \tag{66}$$

is then the induced representation

$$\Gamma(\hat{q}) = q + \mathrm{i}\hbar\frac{\partial}{\partial p} \qquad \Gamma(\hat{p}) = -\mathrm{i}\hbar\frac{\partial}{\partial q} \qquad \Gamma(\hat{I}) = 1. \tag{67}$$

This representation is obtained in geometric quantization starting with the Poisson bracket of any two functions $\mathcal{A}$ and $\mathcal{B}$ in the HW algebra in the form

$$\{\mathcal{A}, \mathcal{B}\} = \frac{\partial\mathcal{A}}{\partial q}\frac{\partial\mathcal{B}}{\partial p} - \frac{\partial\mathcal{A}}{\partial p}\frac{\partial\mathcal{B}}{\partial q}. \tag{68}$$

Corresponding vector fields are then defined by

$$X_{\mathcal{Q}} = \frac{\partial}{\partial p} \qquad X_{\mathcal{P}} = -\frac{\partial}{\partial q} \qquad X_{\mathcal{I}} = 0 \tag{69}$$

and a symplectic form, for which

$$\{\mathcal{A}, \mathcal{B}\} = \omega(X_{\mathcal{A}}, X_{\mathcal{B}}) \tag{70}$$

is given by

$$\omega = \mathrm{d}q \wedge \mathrm{d}p. \tag{71}$$

This two-form is exact and expressible as the exterior derivative $\omega = \mathrm{d}\theta$ of a variety of one-forms. For the (p, q) coordinates defined by setting $T(g(p,q)) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}q\hat{p}}$, cf equation (65), the identities

$$\begin{aligned} \hat{q}T(g(p,q)) &= \mathrm{i}\hbar\frac{\partial}{\partial p}T(g(p,q)) \\ \hat{p}T(g(p,q)) &= \left(-\mathrm{i}\hbar\frac{\partial}{\partial q} + p\right)T(g(p,q)) \end{aligned} \tag{72}$$

imply that an appropriate one-form is

$$\theta = -p\,\mathrm{d}q. \tag{73}$$

Prequantization of the $\{\mathcal{Q}, \mathcal{P}, \mathcal{I}\}$ functions then gives

$$\begin{aligned} \Gamma(\mathcal{Q}) &= \mathcal{Q} + \mathrm{i}\hbar X_{\mathcal{Q}} - \theta(X_{\mathcal{Q}}) = q + \mathrm{i}\hbar\frac{\partial}{\partial p} \\ \Gamma(\mathcal{P}) &= \mathcal{P} + \mathrm{i}\hbar X_{\mathcal{P}} - \theta(X_{\mathcal{P}}) = -\mathrm{i}\hbar\frac{\partial}{\partial q} \\ \Gamma(\mathcal{I}) &= \mathcal{I} - X_{\mathcal{I}} - \theta(X_{\mathcal{I}}) = 1 \end{aligned} \tag{74}$$

which is identical to the induced representation of equation (67).

To obtain an irreducible representation, a functional $\langle\varphi|$ on a suitably-defined dense subspace $\mathbb{H}_D \subset \mathbb{H}$ may be chosen such that

$$\langle\varphi|\hat{q}|\psi\rangle = 0 \qquad \forall|\psi\rangle \in \mathbb{H}_D. \tag{75}$$

This choice corresponds to choosing the real polarization $\mathfrak{p}$ spanned by the operators $\hat{q}$ and $\hat{I}$. Note that the state $|\varphi\rangle$ is not a normalizable state vector of the Hilbert space of square-integrable functions on the HW group. Nevertheless, the bra vector $\langle\varphi|$ is a well-defined functional on $\mathbb{H}_D$. The coherent state wavefunctions, for states in this dense subspace, are then the p-independent functions, given by

$$\psi(q) = \langle\varphi|\mathrm{e}^{\frac{\mathrm{i}}{\hbar}q\hat{p}}|\psi\rangle \tag{76}$$

and the coherent state representation of the algebra reduces to the familiar irreducible Schrödinger representation

$$\Gamma(\hat{q}) = q \qquad \Gamma(\hat{p}) = -\mathrm{i}\hbar\frac{\partial}{\partial q} \qquad \Gamma(\hat{I}) = 1. \tag{77}$$

4.1.2. The Bargmann–Segal representation. To obtain a classical Bargmann–Segal representation of the HW algebra, choose any normalized state $|0\rangle$ in the Hilbert space such that

$$\langle 0|\hat{a}^\dagger|0\rangle = \langle 0|\hat{a}|0\rangle = 0 \qquad \langle 0|\hat{I}|0\rangle = 1 \tag{78}$$

where

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{q} - \frac{\mathrm{i}}{\hbar}\hat{p}\right) \qquad \hat{a} = \frac{1}{\sqrt{2}}\left(\hat{q} + \frac{\mathrm{i}}{\hbar}\hat{p}\right). \tag{79}$$

With group elements parametrized by the factorization

$$T(g(z, z^*, \varphi)) = \mathrm{e}^{(\mathrm{i}\varphi - \frac{1}{2}|z|^2)\hat{I}}\, \mathrm{e}^{-z^*\hat{a}^\dagger}\, \mathrm{e}^{z\hat{a}} \tag{80}$$

the group conjugates of $\{\hat{a}^\dagger, \hat{a}, \hat{I}\}$ are given by

$$\begin{aligned} \hat{a}^\dagger(z, z^*) &= T(g)\hat{a}^\dagger T(g^{-1}) = \hat{a}^\dagger + z\hat{I} \\ \hat{a}(z, z^*) &= T(g)\hat{a}T(g^{-1}) = \hat{a} + z^*\hat{I} \\ \hat{I}(z, z^*) &= T(g)\hat{I}T(g^{-1}) = \hat{I} \end{aligned} \tag{81}$$

and the corresponding conjugates of $\hat{q}$ and $\hat{p}$ are

$$\begin{aligned} \hat{q}(z, z^*) &= \hat{q} + \frac{1}{\sqrt{2}}(z + z^*)\hat{I} \\ \hat{p}(z, z^*) &= \hat{p} + \frac{\mathrm{i}\hbar}{\sqrt{2}}(z - z^*)\hat{I}. \end{aligned} \tag{82}$$

This parameterization leads to a classical realization of the HW algebra in which $\{\hat{q}, \hat{p}, \hat{I}\}$ map to functions $\{\mathcal{Q}, \mathcal{P}, \mathcal{I}\}$ of z and z^* defined by

$$\begin{aligned} \mathcal{Q}(z, z^*) &= \langle 0|\hat{q}(z, z^*)|0\rangle = \frac{1}{\sqrt{2}}(z + z^*) \\ \mathcal{P}(z, z^*) &= \langle 0|\hat{p}(z, z^*)|0\rangle = \frac{\mathrm{i}\hbar}{\sqrt{2}}(z - z^*) \\ \mathcal{I}(z, z) &= \langle 0|\hat{I}(z, z^*)|0\rangle = 1 \end{aligned} \tag{83}$$

and with Poisson bracket given, for example, by

$$\{\mathcal{Q}, \mathcal{P}\}(z, z^*) = -\frac{\mathrm{i}}{\hbar}\langle 0|[\hat{q}(z, z^*), \hat{p}(z, z^*)]|0\rangle = \mathcal{I}(z, z^*). \tag{84}$$

With $\langle\varphi| = \langle 0|$, a state $|\psi\rangle$ is now assigned a coherent state wavefunction ψ defined over the complex z plane by

$$\psi(z, z^*) = \langle\varphi|\mathrm{e}^{-z^*\hat{a}^\dagger}\, \mathrm{e}^{z\hat{a}}|\psi\rangle. \tag{85}$$

The corresponding coherent state representation Γ of an element $\hat{A}$ of the Heisenberg–Weyl algebra, defined generally by

$$[\Gamma(\hat{A})\psi](x) = \langle\varphi|\mathrm{e}^{-z^*\hat{a}^\dagger}\, \mathrm{e}^{z\hat{a}}\hat{A}|\psi\rangle = \langle\varphi|\mathrm{e}^{-z^*\hat{a}^\dagger}(\hat{A} + z[\hat{a}, \hat{A}])\,\mathrm{e}^{z\hat{a}}|\psi\rangle \tag{86}$$

then gives the prequantization

$$\Gamma(a) = \frac{\partial}{\partial z} \qquad \Gamma(a^\dagger) = z - \frac{\partial}{\partial z^*} \qquad \Gamma(I) = 1. \tag{87}$$

To obtain an irreducible representation, define $|\varphi\rangle$ to be the vacuum state for which

$$\hat{a}|\varphi\rangle = 0 \qquad \hat{I}|\varphi\rangle = |\varphi\rangle. \tag{88}$$

This state satisfies the equation

$$\langle\varphi|\hat{I}|\psi\rangle = \langle\varphi|\psi\rangle \qquad \langle\varphi|\hat{a}^\dagger|\psi\rangle = 0 \tag{89}$$

and defines a complex polarization $\mathfrak{p} \subset \mathfrak{g}^c$ spanned by the operators $\hat{a}^\dagger$ and $\hat{I}$. The coherent state wavefunctions are now the holomorphic functions, given for $|\psi\rangle$ in the dense subspace of $\mathbb{H}$ generated by the action of finite powers of $a^\dagger$ on the vacuum state by

$$\psi(z) = \langle\varphi|\mathrm{e}^{z\hat{a}}|\psi\rangle. \tag{90}$$

The corresponding coherent state representation of the HW algebra is now the well-known Bargmann–Segal representation

$$\Gamma(a) = \frac{\partial}{\partial z} \qquad \Gamma(a^\dagger) = z \qquad \Gamma(I) = 1 \tag{91}$$

which is known to be irreducible.

The Hilbert $\mathcal{H}$ space for this irrep is inferred in coherent state theory from the requirement that $\partial/\partial z$ should be the Hermitian adjoint of z for a unitary representation. Thus, $\mathcal{H}$ has an orthonormal basis $\{\psi_n; n = 0, 1, 2, \ldots\}$ with

$$\psi_n(z) = \frac{z^n}{\sqrt{n!}} \tag{92}$$

and inner product

$$(\psi_m, \psi_n) = \frac{1}{\sqrt{m!n!}} \left(\frac{\partial^m}{\partial z^m} z^n\right)\bigg|_{z=0} = \delta_{mn}. \tag{93}$$

From the observation that

$$\int \left(\frac{\partial\psi}{\partial z}\right)^* \mathrm{e}^{-zz^*} \psi'(z)\,\mathrm{d}z\,\mathrm{d}z^* = \int \psi(z)^*\,\mathrm{e}^{-zz^*} z\psi'(z)\,\mathrm{d}z\,\mathrm{d}z^* \tag{94}$$

it is also determined that $\mathcal{H}$ is the space of holomorphic functions with norm

$$(\psi, \psi) = \frac{1}{2\pi} \int |\psi(z)|^2\,\mathrm{e}^{-zz^*}\,\mathrm{d}z\,\mathrm{d}z^*. \tag{95}$$

This Hilbert space $\mathcal{H}$ is the well-known Bargmann–Segal space of entire analytic functions.

4.2. The semisimple su(2) algebra

Representations of the groups $SO(3)$ and $SO(2, 1)$ were constructed within the framework of Kirillov's orbit method by Dunne [23]. We consider here only $su(2)$ (isomorphic to $so(3)$) as an example of a semisimple Lie algebra.

Suppose the regular representation T of the $su(2)$ algebra is spanned by three components of angular momentum $(\hat{S}_1, \hat{S}_2, \hat{S}_3)$ with commutation relations

$$[\hat{S}_i, \hat{S}_j] = \mathrm{i}\hat{S}_k \qquad i, j, k \text{ cyclic} \tag{96}$$

acting on $\mathcal{L}^2(SU(2))$.

Elements of the $SU(2)$ group can be parametrized in many ways. The standard parameterization, in terms of Euler angles,

$$T(g(\alpha, \beta, \gamma)) = \mathrm{e}^{-\mathrm{i}\alpha\hat{S}_3}\,\mathrm{e}^{-\mathrm{i}\beta\hat{S}_2}\,\mathrm{e}^{-\mathrm{i}\gamma\hat{S}_3} \tag{97}$$

leads to a classical realization of the $su(2)$ Lie algebra and a prequantization. However, with this parameterization, it is not so easy to identify a polarization and an irreducible subrepresentation. Parameterizations that lead naturally to irreducible quantizations are defined as follows.

4.2.1. Representation by functions on a circle. Because the Lie algebras $su(2)$ and $sl(2,\mathbb{R})$ are both real forms of $sl(2,\mathbb{C})$, it follows that the irreps of $su(2)$ are defined by corresponding finite-dimensional irreps of $sl(2,\mathbb{R})$. Thus, it is useful to regard the operators $\{\hat{S}_1, \mathrm{i}\hat{S}_2, \hat{S}_3\}$ as spanning a finite-dimensional irrep of $sl(2,\mathbb{R})$ and use the Iwasawa factorization to represent an element $g \in SL(2,\mathbb{R})$ in the parametrized form

$$T(g(y,z,\theta)) = \mathrm{e}^{y\hat{S}_3}\,\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2} \tag{98}$$

where $\hat{S}_- = \hat{S}_1 - \mathrm{i}\hat{S}_2$. Let $|0\rangle \in \mathcal{L}^2(SU(2))$ be a normalized state such that

$$\langle 0|\hat{S}_3|0\rangle = M \qquad \langle 0|\hat{S}_1|0\rangle = \langle 0|\hat{S}_2|0\rangle = 0 \tag{99}$$

where M is real. We then obtain the classical realization of the $su(2)$ algebra, $\hat{S}_i \to \mathcal{S}_i$, as functions on a cylinder

$$\begin{aligned}
\mathcal{S}_1(z,\theta) &= \langle 0|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\,\hat{S}_1\,\mathrm{e}^{-\mathrm{i}\theta\hat{S}_2}\,\mathrm{e}^{-z\hat{S}_-}|0\rangle = M(\sin\theta - z\cos\theta)\\
\mathcal{S}_2(z,\theta) &= \langle 0|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\,\hat{S}_2\,\mathrm{e}^{-\mathrm{i}\theta\hat{S}_2}\,\mathrm{e}^{-z\hat{S}_-}|0\rangle = \mathrm{i}Mz\\
\mathcal{S}_3(z,\theta) &= \langle 0|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\,\hat{S}_3\,\mathrm{e}^{-\mathrm{i}\theta\hat{S}_2}\,\mathrm{e}^{-z\hat{S}_-}|0\rangle = M(\cos\theta + z\sin\theta).
\end{aligned} \tag{100}$$

The Poisson bracket of these functions

$$\{\mathcal{S}_i,\mathcal{S}_j\}(z,\theta) = -\mathrm{i}\langle 0|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}[\hat{S}_i,\hat{S}_j]\,\mathrm{e}^{-\mathrm{i}\theta\hat{S}_2}\,\mathrm{e}^{-z\hat{S}_-}|0\rangle \tag{101}$$

gives

$$\{\mathcal{S}_i,\mathcal{S}_j\} = \mathcal{S}_k \qquad i,j,k \text{ cyclic}. \tag{102}$$

It can also be expressed in the classical form

$$\{\mathcal{S}_i,\mathcal{S}_j\}(z,\theta) = \frac{\mathrm{i}}{M}\left(\frac{\partial\mathcal{S}_i}{\partial z}\frac{\partial\mathcal{S}_j}{\partial\theta} - \frac{\partial\mathcal{S}_i}{\partial\theta}\frac{\partial\mathcal{S}_j}{\partial z}\right). \tag{103}$$

The quantizability condition is that $2M$ should be an integer. Prequantization is then given by choosing $|\varphi\rangle$ to be an eigenstate of $\hat{S}_3$ with eigenvalue M (a half integer) so that

$$\langle\varphi|\mathrm{e}^{\mathrm{i}\sigma\hat{S}_3}\,\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}|\psi\rangle = \mathrm{e}^{\mathrm{i}M\sigma}\langle\varphi|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}|\psi\rangle. \tag{104}$$

Coherent state wavefunctions for the induced representation are now defined by

$$\psi(z,\theta) = \langle\varphi|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}|\psi\rangle \qquad |\psi\rangle \in \mathcal{L}^2(SU(2)) \tag{105}$$

and the corresponding representation of the infinitesimal generators of $SU(2)$ is defined in the usual way by

$$[\Gamma(S_i)\psi](z,\theta) = \langle\varphi|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\,\hat{S}_i|\psi\rangle. \tag{106}$$

This equation gives $\Gamma(S_2)$ immediately as

$$\Gamma(S_2) = -\mathrm{i}\frac{\partial}{\partial\theta}. \tag{107}$$

From

$$\begin{aligned}
[\Gamma(S_1)\psi](z,\theta) &= \langle\varphi|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\,\hat{S}_1|\psi\rangle\\
&= \langle\varphi|\mathrm{e}^{z\hat{S}_-}[\hat{S}_1\cos\theta + \hat{S}_3\sin\theta]\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}|\psi\rangle
\end{aligned} \tag{108}$$

$$[\Gamma(S_3)\psi](z,\theta) = \langle\varphi|\mathrm{e}^{z\hat{S}_-}[\hat{S}_3\cos\theta - \hat{S}_1\sin\theta]\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}|\psi\rangle \tag{109}$$

and the observation that

$$\mathrm{e}^{z\hat{S}_-}\hat{S}_1 = \mathrm{e}^{z\hat{S}_-}(\hat{S}_- + \mathrm{i}\hat{S}_2) \qquad \mathrm{e}^{z\hat{S}_-}\hat{S}_3 = [\hat{S}_3 + z\hat{S}_-]\mathrm{e}^{z\hat{S}_-} \tag{110}$$

it follows that

$$\begin{aligned}\Gamma(S_1) &= \sin\theta\left[M + z\frac{\partial}{\partial z}\right] + \cos\theta\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial\theta}\right]\\ \Gamma(S_3) &= \cos\theta\left[M + z\frac{\partial}{\partial z}\right] - \sin\theta\left[\frac{\partial}{\partial z} + \frac{\partial}{\partial\theta}\right].\end{aligned} \tag{111}$$

An irreducible subrepresentation results if $|\varphi\rangle$ is chosen to be a highest weight state, so that $2M$ is a positive integer (which we now call $2S$), and satisfies the equations

$$\hat{S}_3|\varphi\rangle = S|\varphi\rangle \qquad \hat{S}_+|\varphi\rangle = 0 \tag{112}$$

where

$$\hat{S}_+ = \hat{S}_1 + \mathrm{i}\hat{S}_2 \tag{113}$$

is the adjoint of $\hat{S}_-$. The coherent state wavefunctions then become independent of z,

$$\psi(z,\theta) = \langle\varphi|\mathrm{e}^{z\hat{S}_-}\,\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}|\psi\rangle = \langle\varphi|\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}|\psi\rangle \tag{114}$$

and are seen to be functions on the circle. The coherent state representation reduces to

$$\begin{aligned}\Gamma(\hat{S}_2) &= -\mathrm{i}\frac{\partial}{\partial\theta}\\ \Gamma(\hat{S}_1) &= S\sin\theta + \cos\theta\frac{\partial}{\partial\theta}\\ \Gamma(\hat{S}_3) &= S\cos\theta - \sin\theta\frac{\partial}{\partial\theta}\end{aligned} \tag{115}$$

which is that of an $su(2)$ irrep of angular momentum S. It corresponds to the quantization obtained by choosing the polarization $\mathfrak{p}$ to be the Borel subalgebra of $su(2)^c$ spanned by S_3 and S_-.

The inner product and Hilbert space for this irrep are inferred in coherent state theory from the requirement that, for a unitary representation, $\Gamma(S_+)$ should be the Hermitian adjoint of $\Gamma(S_-)$ and vice versa. Thus, an orthonormal basis $\mathcal{H}$ is constructed by the systematic methods of K-matrix theory [33]. The inner product is also given in integral form by the methods of [34].

4.2.2. Representation by holomorphic functions. Equivalent holomorphic representations are obtained by choosing the same polarization but a different factorization of an $SL(2,\mathbb{R})$ group element

$$T(g(x,y,z)) = \mathrm{e}^{x\hat{S}_3}\,\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}. \tag{116}$$

Then, for $|0\rangle$ again such that equation (99) is satisfied, we obtain a classical realization of the $su(2)$ algebra, $\hat{S}_i \to \mathcal{S}_i$, with

$$\begin{aligned}\mathcal{S}_1(y,z) &= \langle 0|\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}\hat{S}_1\,\mathrm{e}^{-z\hat{S}_+}\,\mathrm{e}^{-y\hat{S}_-}|0\rangle = M(z - y + yz^2)\\ \mathcal{S}_2(y,z) &= \langle 0|\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}\hat{S}_2\,\mathrm{e}^{-z\hat{S}_+}\,\mathrm{e}^{-y\hat{S}_-}|0\rangle = \mathrm{i}M(z + y + yz^2)\\ \mathcal{S}_3(y,z) &= \langle 0|\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}\hat{S}_3\,\mathrm{e}^{-z\hat{S}_+}\,\mathrm{e}^{-y\hat{S}_-}|0\rangle = M(1 + 2yz).\end{aligned} \tag{117}$$

The Poisson bracket of these functions

$$\{\mathcal{S}_i,\mathcal{S}_j\}(y,z) = -\mathrm{i}\langle 0|\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}[\hat{S}_i,\hat{S}_j]\,\mathrm{e}^{-z\hat{S}_+}\,\mathrm{e}^{-y\hat{S}_-}|0\rangle \tag{118}$$

again gives

$$\{\mathcal{S}_i,\mathcal{S}_j\} = \mathcal{S}_k \qquad i,j,k \text{ cyclic} \tag{119}$$

and is now expressed in the classical form

$$\{\mathcal{S}_i, \mathcal{S}_j\}(z, \theta) = \frac{\mathrm{i}}{2M}\left(\frac{\partial \mathcal{S}_i}{\partial y}\frac{\partial \mathcal{S}_j}{\partial z} - \frac{\partial \mathcal{S}_i}{\partial z}\frac{\partial \mathcal{S}_j}{\partial y}\right). \tag{120}$$

With $|\varphi\rangle$ an eigenstate of $\hat{S}_3$ of eigenvalue M (with $2M$ an integer), coherent state wavefunctions are given by

$$\psi(y, z) = \langle\varphi|\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}|\psi\rangle \qquad |\psi\rangle \in \mathcal{L}^2(SU(2)) \tag{121}$$

and the corresponding representation of the infinitesimal generators of $SU(2)$ is defined in the usual way by

$$[\Gamma(S_i)\psi](y, z) = \langle\varphi|\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}\hat{S}_i|\psi\rangle. \tag{122}$$

One finds that

$$\begin{aligned}
\Gamma(S_+) &= \frac{\partial}{\partial z} \\
\Gamma(S_-) &= \frac{\partial}{\partial y} + z\left(2M + 2y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}\right) \\
\Gamma(S_3) &= M + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}.
\end{aligned} \tag{123}$$

This representation is also obtained by prequantization of the above classical realization.

An irreducible subrepresentation is again obtained by requiring $|\varphi\rangle$ to be a highest weight state satisfying equation (112). The coherent state wavefunctions then become independent of y,

$$\psi(z) = \langle\varphi|\mathrm{e}^{y\hat{S}_-}\,\mathrm{e}^{z\hat{S}_+}|\psi\rangle = \langle\varphi|\mathrm{e}^{z\hat{S}_+}|\psi\rangle \tag{124}$$

and holomorphic functions of the variable z. The coherent state representation reduces to

$$\Gamma(S_+) = \frac{\partial}{\partial z} \qquad \Gamma(S_-) = z\left(2S - z\frac{\partial}{\partial z}\right) \qquad \Gamma(S_3) = S - z\frac{\partial}{\partial z} \tag{125}$$

which is that of an $su(2)$ irrep of angular momentum S. It corresponds to the quantization obtained by choosing the polarization $\mathfrak{p}$ to be the Borel subalgebra of $su(2)^c$ spanned by S_3 and S_-.

Note that the last two examples involve the same polarization, but give different realizations.

4.3. The semidirect product Euclidean group in two dimensions

The Euclidean group in two dimensions $E(2) \sim [\mathbb{R}^2]SO(2)$ can be realized as the group of translations and rotations in a real two-dimensional Euclidean space. Its infinitesimal generators are two components (p_x, p_y) of a momentum vector and an angular momentum L. Alternatively, it can be realized as the dynamical group of a two-dimensional rotor, e.g., a particle moving in a circle. A set of observables for such a rotor is given by a pair of (x, y) coordinate functions and again an angular momentum. Let T be the regular $E(2)$ representation, with observables in the algebra satisfying the commutation relations

$$[\hat{x}, \hat{y}] = 0 \qquad [\hat{L}, \hat{x}] = \mathrm{i}\hbar\hat{y} \qquad [\hat{L}, \hat{y}] = -\mathrm{i}\hbar\hat{x}. \tag{126}$$

Group elements in $E(2)$ can be parametrized as

$$T(g(\alpha,\beta,\theta)) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\alpha\hat{x}+\beta\hat{y})}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}. \tag{127}$$

Let $|0\rangle$ be a state in the Hilbert space of T having expectation values

$$\langle 0|\hat{x}|0\rangle = 0 \qquad \langle 0|\hat{y}|0\rangle = r \qquad \langle 0|\hat{L}|0\rangle = 0. \tag{128}$$

This state defines a classical realization of the $E(2)$ Lie algebra in which (x, y, L) map to β-independent functions $(\mathcal{X}, \mathcal{Y}, \mathcal{L})$ over $g(\alpha,\beta,\theta)$ defined by

$$\begin{aligned}
\mathcal{X}(\alpha,\theta) &= \langle 0|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\alpha\hat{x}+\beta\hat{y})}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}\,\hat{x}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\theta\hat{L}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\alpha\hat{x}+\beta\hat{y})}|0\rangle = r\sin\theta\\
\mathcal{Y}(\alpha,\theta) &= \langle 0|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\alpha\hat{x}+\beta\hat{y})}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}\,\hat{y}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\theta\hat{L}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\alpha\hat{x}+\beta\hat{y})}|0\rangle = r\cos\theta\\
\mathcal{L}(\alpha,\theta) &= \langle 0|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(\alpha\hat{x}+\beta\hat{y})}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}\,\hat{L}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\theta\hat{L}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\alpha\hat{x}+\beta\hat{y})}|0\rangle = -\alpha r
\end{aligned} \tag{129}$$

and for which the Poisson bracket is given by

$$\{\mathcal{A},\mathcal{B}\}(\alpha,\theta) = \frac{1}{r}\left(\frac{\partial\mathcal{A}}{\partial\theta}\frac{\partial\mathcal{B}}{\partial\alpha} - \frac{\partial\mathcal{A}}{\partial\alpha}\frac{\partial\mathcal{B}}{\partial\theta}\right). \tag{130}$$

Now let $\langle\varphi|$ be a functional on a dense subspace $\mathbb{H}_D$ of the Hilbert space for the representation T such that

$$\langle\varphi|T(g(\alpha,\beta,\theta))|\psi\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\beta r}\langle\varphi|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\alpha\hat{x}}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}|\psi\rangle. \tag{131}$$

The space of coherent state wavefunctions, defined for each $|\psi\rangle \in \mathbb{H}_D$ by

$$\psi(\alpha,\theta) = \langle\varphi|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\alpha\hat{x}}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}|\psi\rangle \tag{132}$$

is then isomorphic to the space of square-integrable functions on a cylinder with respect to the standard $\mathrm{d}\alpha\,\mathrm{d}\theta$ measure. This space carries a reducible representation of $E(2)$ for which

$$\begin{aligned}
\Gamma(x) &= r\sin\theta + \mathrm{i}\hbar\cos\theta\frac{\partial}{\partial\alpha}\\
\Gamma(y) &= r\cos\theta - \mathrm{i}\hbar\sin\theta\frac{\partial}{\partial\alpha}\\
\Gamma(L) &= \mathrm{i}\hbar\frac{\partial}{\partial\theta}.
\end{aligned} \tag{133}$$

This representation is the same as that obtained for $E(2)$ by prequantization of the rotor.

To obtain an irreducible representation, one must choose a subalgebra that in geometric quantization defines a polarization. A suitable subalgebra is the Lie algebra of the normal subgroup $\mathbb{R}^2 \subset E(2)$. Let $\langle\varphi|$ be a functional on a dense subspace of the Hilbert space of T such that

$$\langle\varphi|T(\alpha,\beta,\theta)|\psi\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}r\beta}\langle\varphi|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}|\psi\rangle. \tag{134}$$

The space of coherent state wavefunctions, defined for each $|\psi\rangle \in \mathbb{H}_D$ by

$$\psi(\theta) = \langle\varphi|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\theta\hat{L}}|\psi\rangle \tag{135}$$

is now $\mathcal{L}^2(SO(2))$, the space of square-integrable functions on the circle with respect to the standard $\mathrm{d}\theta$ measure. The coherent state representation of the $E(2)$ Lie algebra is now irreducible on $\mathcal{L}^2(SO(2))$ and given by

$$\Gamma(x) = r\sin\theta \qquad \Gamma(y) = r\cos\theta \qquad \Gamma(L) = \mathrm{i}\hbar\frac{\partial}{\partial\theta}. \tag{136}$$

5. Concluding remarks

The theory of geometric quantization provides a sophisticated perspective on the underlying principles for quantization of a classical model. On the other hand, the theory of induced representations is one of the most versatile procedures for constructing representations of Lie groups and Lie algebras. As emphasized by researchers in both fields, the two theories have much in common and both contribute substantially to the description of quantum systems. Unfortunately, because of their formidable mathematical expressions, they are not readily accessible to most physicists. Thus, it is useful to know that both theories can be expressed in the language of coherent state representations, a language that has been specifically developed to provide practical methods for performing algebraic calculations in physics.

It has been shown in this paper that the coherent state construction yields the three types of representations of the SGA of an algebraic model involved in a quantization scheme: classical realizations, prequantizations and full quantizations (unitary irreps). Examples have also been given to illustrate how the coherent state approach provides an intuitive path through the techniques of geometric quantization for algebraic models. Thus, we are optimistic that the coherent state methods presented here will serve to make the methods of induced representations and geometric quantization accessible to a wider community. By expressing the methods of geometric quantization in the language of coherent state theory, we are also optimistic that many techniques developed for the practical application of induced representation theory to the solution of physical problems will be equally useful for practical applications of the methods of geometric quantization.

It is interesting that the classical representations given by coherent state methods automatically take into account the inherent limitations, imposed by the uncertainty principle, on an experimentalist's ability to measure an observable precisely.

In the conventional interpretation of quantum mechanics, the expectation $\mathcal{X} = \langle\psi|\hat{X}|\psi\rangle$ of an observable is identified with the mean value of many (precise) measurements of the value of the observable when the system is in a state $|\psi\rangle$. Thus, the distribution of experimental values, using an ideal measurement in which the only limitations on accuracy are quantum mechanical, is given by the variance

$$\sigma^2(\mathcal{X}) = \langle\psi|(\hat{X} - \mathcal{X})^2|\psi\rangle. \tag{137}$$

It is remarkable then that the mean expectation values of the observables of an SGA define functions on a coadjoint orbit which are precisely those of a classical representation.

It is also noted that a given unitary irrep of an SGA can give rise to many classical realizations. For example, if a freely rotating object had squared angular momentum given by $L(L+1)$ in a unitary $SO(3)$ irrep, then the corresponding classical value given by $\sum_k \mathcal{L}_k^2$ with $\mathcal{L}_k = \langle\psi|\hat{L}_k|\psi\rangle$ can have values ranging from zero to L^2. The maximum value of L^2 would be obtained when $|\psi\rangle$ is a minimum uncertainty (e.g. a highest weight) state. If an experimentalist could put the object into such a minimal uncertainty state, then he/she would obtain an integer value for L and have determined the quantal state of the rotor precisely. However, in a practical situation, the uncertainties in a given experimental situation inevitably exceed the minimal uncertainties permitted by quantum mechanics.

It is now known that scalar coherent theory has a natural generalization to a vector coherent theory [18] in which an irrep of a Lie algebra is induced from a multidimensional irrep of a subalgebra. In a sequel to this paper we shall show that VCS theory can also be expressed in the language of geometric quantization and that it corresponds to the quantization of a model with intrinsic degrees of freedom.

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Appendix A. The covariant derivative and gauge potential

If $\mathcal{A}$ is a classical representation of an element $A \in \mathfrak{g}$ as a function on $\mathcal{O}_\rho = H_\rho\backslash G$ then the corresponding Hamiltonian vector field $X_\mathcal{A}$ on $\mathcal{O}_\rho$ is defined to operate on a function f on $\mathcal{O}_\rho$ such that $X_\mathcal{A} f$ is equal to the Poisson bracket $\{\mathcal{A}, f\}$. This requirement means that $X_\mathcal{A}$ must satisfy

$$[X_\mathcal{A} f](g) = \sum_{\mu\nu} \partial_\mu \mathcal{A}(g)\, \omega^{\mu\nu}\, \partial_\nu f(g) = \sum_\nu A^\nu(g)\, \partial_\nu \mathcal{B}(g), \tag{A.1}$$

where $A^\nu(g)$ is a coefficient in the expansion

$$A(g) \equiv \mathrm{Ad}_g(A) = \sum_i A^i(g) A_i + \sum_\nu A^\nu(g) A_\nu \tag{A.2}$$

and $\partial_\nu f$ is defined by

$$\partial_\nu f(g) = \frac{\partial}{\partial x^\nu} f\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\sum_\alpha x^\alpha A_\alpha} g\right)\Big|_{x=0}. \tag{A.3}$$

Claim. *Let $\mathcal{A}$ be a classical representation of an element $A \in \mathfrak{g}$ as a function on $\mathcal{O}_\rho = H_\rho\backslash G$ and $X_\mathcal{A}$ the corresponding Hamiltonian vector field on $\mathcal{O}_\rho$. Then the covariant derivative $\nabla_{X_\mathcal{A}}$ (cf equation (48)) acts on a coherent state wavefunction by*

$$[\nabla_{X_\mathcal{A}} \psi](g) = \sum_\nu A^\nu(g) \left(\partial_\nu + \frac{\mathrm{i}}{\hbar}\rho(A_\nu)\right)\psi(g), \tag{A.4}$$

It is expressible in the form

$$\nabla_{X_\mathcal{A}} = X_\mathcal{A} + \frac{\mathrm{i}}{\hbar}\theta(X_\mathcal{A}), \tag{A.5}$$

where θ is a symplectic potential (one-form) for $\mathcal{O}_\rho$.

Before proving this claim, we consider first the expression of the vector field $X_\mathcal{A}$ as a derivation in terms of local coordinates by means of the following observation.

Observation. *If $X(x) = -\frac{\mathrm{i}}{\hbar}\sum_\mu x^\mu A_\mu$, then*

$$\mathrm{i}\hbar\, \frac{\partial \mathrm{e}^{X(x)}}{\partial x^\nu} = A_\nu(x)\mathrm{e}^{X(x)} = \mathrm{e}^{X(x)} A_\nu(-x), \tag{A.6}$$

and

$$\mathrm{i}\hbar\, \frac{\partial \mathrm{e}^{-X(x)}}{\partial x^\nu} = -A_\nu(-x)\mathrm{e}^{-X(x)} = -\mathrm{e}^{-X(x)} A_\nu(x), \tag{A.7}$$

where

$$A_\nu(x) = A_\nu + \frac{1}{2!}[X(x), A_\nu] + \frac{1}{3!}[X(x), [X(x), A_\nu]] + \cdots \tag{A.8}$$

Proof. The first identity of equation (A.6) follows from the observation that

$$\frac{\partial}{\partial x^\nu}\left(\mathrm{e}^{-X(x)}\mathrm{e}^{X(x)}\right) = 0, \tag{A.9}$$

and hence that

$$\mathrm{i}\hbar\frac{\partial}{\partial x^\nu}\mathrm{e}^{X(x)} = \mathrm{i}\hbar\left(\frac{\partial}{\partial x^\nu} - \mathrm{e}^{X(x)}\frac{\partial}{\partial x^\nu}\mathrm{e}^{-X(x)}\right)\mathrm{e}^{X(x)}. \tag{A.10}$$

The second identity of equation (A.6) is obtained directly from

$$\frac{\partial}{\partial x^\nu}\mathrm{e}^{X(x)} = \mathrm{e}^{X(x)}\left(\mathrm{e}^{-X(x)}\frac{\partial}{\partial x^\nu}\mathrm{e}^{X(x)}\right). \tag{A.11}$$

Equation (A.7) is obtained similarly. □

If $g(x) = \mathrm{e}^{X(x)}g$ then it follows from the observation that to leading order in δx,

$$g(x+\delta x) = \exp\left[-\frac{i}{\hbar}\sum_\mu \delta x^\nu A_\nu(x)\right]g(x). \tag{A.12}$$

Hence, with the expansion

$$A_\nu(x) = \sum_\mu \Lambda_\nu^\mu(x)A_\mu + \sum_i \lambda_\nu^i(x)A_i, \tag{A.13}$$

the observation leads to the expression for the derivatives of a function f on G

$$\frac{\partial}{\partial x^\nu}f(g(x)) = \sum_\mu \Lambda_\nu^\mu(x)\,\partial_\mu f(g(x)) + \sum_i \lambda_\nu^i(x)\,\partial_i f(g(x)). \tag{A.14}$$

However, when $f = \mathcal{B}$ (a classical observable in the SGA), the second term on the right is zero, due to the fact that $\mathcal{B}(g)$ satisfies the equations $\mathcal{B}(hg) = \mathcal{B}(g)$, and hence

$$\partial_i\,\mathcal{B}(g) = 0. \tag{A.15}$$

Thus, we obtain

$$\partial_\nu\mathcal{B}(g(x)) = \sum_\mu \overline{\Lambda}_\nu^\mu(x)\frac{\partial}{\partial x^\mu}\mathcal{B}(g(x)) \tag{A.16}$$

and

$$[X_\mathcal{A}\mathcal{B}](g(x)) = \sum_{\mu\nu} A^\nu(g(x))\overline{\Lambda}_\nu^\mu(x)\frac{\partial}{\partial x^\mu}\mathcal{B}(g(x)), \tag{A.17}$$

where $\overline{\Lambda}(x)$ is the inverse of the matrix $\Lambda(x)$.

Proof of the Claim. The action of $X_\mathcal{A}$, defined by equation (A.17) can be extended to coherent state wavefunctions. However, while the action (A.17) of $X_\mathcal{A}$ on functions over $\mathcal{O}_\rho \sim H_\rho\backslash G$ is covariant, the action on coherent state wavefunctions is not covariant; these wavefunctions are not defined on $\mathcal{O}_\rho$ but have extra phase factors and, as a consequence, they are not gauge invariant. In particular, $\partial_i\psi(g)$ is not generally zero. Thus, in evaluating the covariant derivative, defined by equation (A.4), both terms on the right hand side of equation (A.14) must be included to give

$$\partial_\nu\psi(g(x)) = \sum_\mu \overline{\Lambda}_\nu^\mu(x)\left(\frac{\partial}{\partial x^\mu} + \frac{\mathrm{i}}{\hbar}\sum_i \lambda_\mu^i\rho(A_i)\right)\psi(g(x)). \tag{A.18}$$

The covariant derivative of ψ at $g(x)$ is then

$$[\nabla_{X_\mathcal{A}}\psi](g(x)) = \sum_\nu A^\nu(g(x))\overline{\Lambda}_\nu^\mu(x)\left(\frac{\partial}{\partial x^\mu} + \frac{\mathrm{i}}{\hbar}\theta_\mu(x)\right)\psi(g(x)), \tag{A.19}$$

where

$$\theta_\mu(x) = \sum_\nu \Lambda_\mu^\nu(x)\rho(A_\nu) + \sum_i \lambda_\mu^i(x)\rho(A_i) = \rho(A_\mu(x)). \tag{A.20}$$

Thus, with the interpretation of θ_ν as a component of a one-form $\theta_{g(x)} = \sum_\nu \theta_\nu(x)\,\mathrm{d}x^\nu$, so that

$$\theta_\nu(x) = \theta_{g(x)}(\partial/\partial x^\nu), \tag{A.21}$$

the covariant derivative is expressed $\nabla_{X_\mathcal{A}} = X_\mathcal{A} + \frac{\mathrm{i}}{\hbar}\theta(X_\mathcal{A})$ as claimed.

It remains to be shown that the one-form θ is a symplectic potential, i.e., that the symplectic form on $\mathcal{O}_\rho$ is given by

$$\omega_{g(x)} = \mathrm{d}\theta_{g(x)} = \sum_{\mu\nu} \frac{\partial\theta_\nu(x)}{\partial x^\mu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{A.22}$$

First observe, from equation (A.20), that

$$\frac{\partial\theta_\nu(x)}{\partial x^\mu} = \rho\left(\frac{\partial A_\nu(x)}{\partial x^\mu}\right). \tag{A.23}$$

Now, with $A_\nu(x)$ written in the form

$$A_\nu(x) = -\mathrm{e}^{X(x)}\mathrm{i}\hbar\,\frac{\partial}{\partial x^\nu}\mathrm{e}^{-X(x)}, \tag{A.24}$$

it follows, from multiple use of the observation, that

$$\frac{\partial A_\nu(x)}{\partial x^\mu} = -\frac{\mathrm{i}}{\hbar}[A_\mu(x), A_\nu(x)]. \tag{A.25}$$

Thus, we obtain

$$\frac{\partial\theta_\nu(x)}{\partial x^\mu} = -\frac{\mathrm{i}}{\hbar}\rho([A_\mu(x), A_\nu(x)]), \tag{A.26}$$

which from the definition of $A_\nu(x)$ (equation (A.13)) and $\omega_{\mu\nu}$ (equation (28)) gives

$$\frac{\partial\theta_\nu(x)}{\partial x^\mu} = \sum_{\mu',\nu'} \Lambda_\mu^{\mu'}(x)\omega_{\mu',\nu'}\Lambda_\nu^{\nu'}(x). \tag{A.27}$$

Therefore, if ω is the two-form defined by equation (A.22), then for the vector fields defined by equation (A.17), we obtain

$$\omega_{g(x)}(X_\mathcal{A}, X_\mathcal{B}) = \sum_{\mu\nu} A^\mu(g(x))\,\omega_{\mu\nu}B^\nu(g(x)), \tag{A.28}$$

which is identical to the expression in equation (29) for the Poisson bracket $\{\mathcal{A}, \mathcal{B}\}(g(x))$. This result confirms that the one-form θ is indeed a symplectic potential and completes the proof of the claim. □

References

[1] Kirillov A A 1962 Unitary representations of nilpotent Lie groups *Russian Math. Surveys* **17** 57–110
Kirillov A A 1962 *Usp. Mat. Nauk.* **106** 57

[2] Kostant B 1970 On certain unitary representations which arise from a quantization theory *Group Representations in Mathematics and Physics (Lecture Notes in Physics vol 6)* (Berlin: Springer)
Kostant B 1970 Quantization and unitary representations *Lectures in Modern Analysis and Applications III (Lecture Notes in Mathematics vol 170)* (Berlin: Springer)

[3] Souriau J-M 1966 *Commun. Math. Phys.* **1** 374
Souriau J-M 1970 *Structure des systèmes dynamiques* (Paris: Dunod)

[4] Bartlett S D 2000 Quantization of a classical model with symmetry *Doctoral Thesis* University of Toronto, Toronto, Canada
[5] Mackey G W 1952 *Ann. Math.* **55** 101
Mackey G W 1968 *Induced Representation of Groups and Quantum Mechanics* (New York: Benjamin)
Mackey G W 1978 *Unitary Group Representations in Physics, Probability and Number Theory* (Reading, MA: Benjamin)
[6] Perelomov A 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[7] Klauder J R and Skagerstam B-S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[8] Onofri E 1975 *J. Math. Phys.* **16** 1087
[9] Rowe D J and Repka J 1991 *J. Math. Phys.* **32** 2614
[10] Onofri E and Pauri M 1972 *J. Math. Phys.* **13** 533
Onofri E and Pauri M 1972 *Lett. Nuovov Cimento* **2** 35
[11] Wigner E P 1939 *Ann. Math.* **40** 149
[12] Zak J 1960 *J. Math. Phys.* **1** 165
[13] Goldin G A, Menikoff R and Sharp D H 1987 *Phys. Rev. Lett.* **58** 2162
Goldin G A 1992 *Int. J. Mod. Phys.* B **6** 1905
Penna V and Spera M 1989 *J. Math. Phys.* **30** 2778
Penna V and Spera M 1991 *J. Math. Phys.* **33** 901
[14] Rosensteel G and Ihrig E 1979 *Ann. Phys., NY* **121** 113
[15] Zhao Q 1998 *Commun. Math. Phys.* **194** 135
[16] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)
[17] Glauber R Y 1963 *Phys. Rev.* **131** 2766
[18] Rowe D J 1984 *J. Math. Phys.* **25** 2662
Rowe D J, Rosensteel G and Carr R 1984 *J. Phys. A: Math. Gen.* **17** L399
Rowe D J, Wybourne B G and Butler P H 1985 *J. Phys. A: Math. Gen.* **18** 939
Rowe D J, Rosensteel G and Gilmore R 1985 *J. Math. Phys.* **26** 2787
[19] Rawnsley W 1977 *Quart. J. Math., Oxford* **28** 403
Rawnsley J, Cahen M and Gutt S 1990 *J. Geom. Phys.* **7** 45
[20] Kirillov A A 1975 *Elements of the Theory of Representations* (Berlin: Springer)
[21] Kirillov A 1999 *Bull. Am. Math. Soc.* **36** 433
[22] Streater R F 1967 *Commun. Math. Phys.* **4** 217
[23] Dunne S A 1969 *J. Math. Phys.* **10** 860
[24] Joseph A 1970 *Commun. Math. Phys.* **17** 210
Gotay M J 1995 *Quantization, Coherent States, and Complex Structures (Bialowieza, 1994)* (New York: Plenum)
Gotay M J, Grundling H B and Hurst C A 1996 *Trans. Am. Math. Soc.* **348** 1579
Gotay M J and Grundling H B 1997 *Rep. Math. Phys.* **40** 107
[25] Ihrig E and Rosensteel G 1993 *Int. J. Theor. Phys.* **32** 843
[26] Marsden J E and Ratiu T S 1994 *Introduction to Mechanics and Symmetry* (New York: Springer)
[27] Dirac P A M 1958 *The Principles of Quantum Mechanics* (Oxford: Oxford University Press)
[28] Woodhouse N M J 1991 *Geometric Quantization* (Oxford: Oxford University Press)
[29] Kashiwara M and Vergne M 1978 *Invent. Math.* **44** 1
[30] Wallach N 1973 *Hamonic Analysis on Homogenious Spaces* (New York: Dekker) p 115
[31] Harish-Chandra 1955 *Am. J. Math.* **77** 743
Harish-Chandra 1956 *Am. J. Math.* **78** 1
Harish-Chandra 1956 *Am. J. Math.* **78** 564
[32] Auslander L and Kostant B 1971 *Invent. Math.* **14** 255
[33] Rowe D J, Le Blanc R and Hecht K T 1998 *J. Math. Phys.* **29** 287
Rowe D J 1995 *J. Math. Phys.* **36** 1520
[34] Rowe D J and Repka J in preparation
[35] Bargmann V 1961 *Commun. Pure Appl. Math.* **14** 187
Segal I 1963 *Mathematical Problems of Relativistic Physics* (Providence, RI: Amer. Math. Soc. Publ.)